Principal ideal domains

Def: A _principal ideal domain_ (PID) is an integral domain in which every ideal is principal.

Recall that we showed that all Euclidean domains have this property, so   Fields $\subset$ Euclidean domains $\subset$ PIDs $\subseteq$ Int. domains.

Ex: $\mathbb{Z}$ is a PID, $\mathbb{Z}[x]$ is not (we showed $(2, x)$ is not principal).

Recall that all the nonzero primes in $\mathbb{Z}$ were maximal. This is true in PIDs more generally:

Prop: Every nonzero prime ideal in a PID is maximal.

Pf: Let R be a PID and $(a) \subseteq R$ prime. Suppose $(a) \subseteq (b)$, some $b \in R$.   WTS $(b) = (a)$ or R.

Then $a = rb$, some $r \in R$. Thus, $rb \in (a)$ so $r \in (a)$ or $b \in (a)$. If $b \in (a)$, then $(b) = (a)$.

If $r \in (a)$, then $r = sa \Rightarrow a = sab \Rightarrow (sb - 1)a = 0$, so $sb = 1$, so b is a unit $\Rightarrow (b) = R$. $\square$

We mentioned in the last section that if F is a field, then $F[x]$ is a Euclidean domain, and hence also a PID. The converse is also true.

Intuitively, if $I \subseteq R$ s.t. $I \neq 0$ or R then $(x) \subseteq R[x]$ won't be maximal, since $(x) \subseteq (x, I)$. More precisely:

Corollary: If $R[x]$ is a PID, then R is a field.

Pf: Since $R[x]$ is an integral domain, $R \subseteq R[x]$ must be as well. Thus $R[x]/(x) \cong R$ is an integral domain, so

(x) is prime. Thus since $R[x]$ is a PID, (x) is maximal, so R is a field. $\square$

Ex: $\mathbb{R}[x]$ and $\mathbb{C}[x]$ are both Euclidean domains and hence PIDs.

Consider $(x^2 + 1) \subseteq \mathbb{R}[x]$. We showed on the homework that

$$R[x]/(x^2+1) \cong \mathbb{C}, \text{ so } (x^2+1) \text{ is maximal.}$$

However, in $\mathbb{C}[x]$, $x^2 + 1 = (x+i)(x-i)$, so $(x^2+1) \subseteq (x+i)$ so (x^2+1) is not maximal, and thus not prime.

In fact, in $\mathbb{C}[x]/(x^2+1)$, $\overline{x+i}$ is a zero divisor.